Flat cotorsion modules and the exchange property

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Purity, Approximation Theory and Spectra (PATHS)

Dedicated to Manolo Saorin on the occasion of his 65th birthday.

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Introduction

Let C be a **finitely accessible additive category**:

1. Additive.
2. Has direct limits.
3. Every object is a direct limit of finitely presented objects.
4. C^{fp} is skeletally small.

Relationship with purity and approximation theory

Let C be a **finitely accessible additive category**:

1. Additive.
2. Has direct limits.
3. Every object is a direct limit of finitely presented objects.
4. C^{fp} is skeletally small.

Theorem (Crawley-Boevey)

There exists a skeletally small additive category with split idempotents $\mathcal{S}$ such that

$$\mathsf{C} \text{ is equivalent to } \textit{Flat-}\mathcal{S}$$

- *Mod-*$\mathcal{S}$: The category of additive functors $F : \mathcal{S}^{op} \to Ab$.
- *Flat-*$\mathcal{S}$: The full subcategory of flat functors.

A sequence

$$0 \to A \to B \to C \to 0$$

- Is **pure-exact** in $\mathbf{C}$ if

$$0 \to Hom(F, A) \to Hom(F, B) \to Hom(F, C) \to 0$$

is exact in Ab for any $F \in \mathbf{C}^{fp}$.

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- Is **pure-exact** in $Flat\text{-}\mathcal{S}$ if it is exact in $Mod\text{-}\mathcal{S}$.

The pure-exact structure

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  is exact in *Ab* for any $F \in \mathbf{C}^{fp}$.
- Is **pure-exact** in *Flat*-$\mathcal{S}$ if it is exact in *Mod*-$\mathcal{S}$.

Pure-injectives in *Flat*-$\mathcal{S}$

$M \in$ *Flat*-$\mathcal{S}$ is **pure-injective** if and only if M is **cotorsion** in *Mod*-$\mathcal{S}$.

Cotorsion modules in *Mod*-$\mathcal{S}$

M is cotorsion if and only if $\mathrm{Ext}^1_{\mathcal{S}}(M, F) = 0$ for every $F \in$ *Flat*-$\mathcal{S}$.

We can start with...

Flat cotorsion in module categories.

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Notations

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Objectives

Study the endomorphism ring of **Flat cotorsion modules**:

1. New type of rings: Strong exchange rings.
2. The exchange property.

Right strong exchange rings

This is the starting point of some works by P. A. Guil and I. Herzog such as:

Theorem (Guil-Herzog)

If R is left cotorsion and J is its Jacobson radical, then idempotents lift modulo J, R/J is von Neumann regular (semiregular) and left self-injective.

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- If M is flat and cotorsion, then $\mathrm{End}_R(M)$ is left cotorsion.

Theorem

If M is flat and cotorsion, then $\mathrm{End}_R(M)$ is semiregular and $\mathrm{End}_R(M)/J$ is left self-injective

Modules with semiregular endomorphism ring

1. Quasi-injective modules.
2. Pure-injective modules.
3. Modules with local endomorphism ring.
4. Continuous modules.
5. And now... flat cotorsion

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Answer

They have right strong exchange endomorphism ring.

Right coprime pairs

Right coprime pairs (right comaximal, right unimodular)

A right coprime pair in R is a pair of elements $a, b \in R$ such that $aR + bR = R$. We denote the coprime pair $\langle a, b \rangle$.

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- Equivalence relation between coprime pairs:

$$\langle a, b \rangle \equiv \langle a', b' \rangle \Leftrightarrow aR = a'R, bR = b'R$$

- Preorder relation between coprime pairs:

$$\langle a, b \rangle \leq \langle a', b' \rangle \Leftrightarrow aR \leq a'R, bR \leq b'R$$

Notation

1. Right coprime pair = Equivalence class.
2. $\langle a, b \rangle =$ Equivalence class.
3. $\mathrm{RCP}(R) =$ Set of equivalence classes with the order $\leq$.

Theorem

1. R_R is indecomposable $\Leftrightarrow$ $\mathrm{RCP}(R)$ has exactly two minimal elements.
2. R is local if and only if every coprime pair is trivial, i. e., is of the form $\langle a, u \rangle$ with u a unit.
3. R is left perfect if and only if $\mathrm{RCP}(R)$ has DCC.
4. (Nicholson, Goodearl) R is exchange if and only if for any $\langle a, b \rangle \in \mathrm{RCP}(R)$ there exists a minimal $\langle m, n \rangle$ with $\langle m, n \rangle \leq \langle a, b \rangle$.

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Proof

Nicholson, Goodearl: R is exchange if and only if for any $a \in R$ exists $e^2 = e \in R$ with $eR \leq aR$ and $(1-e)R \leq (1-a)R$.

- $\langle e, 1-e\rangle \leq \langle a, 1-a\rangle$!
- (Herzog-Guil) $\langle e, 1-e\rangle$ is minimal!

Descending chains in RCP(R)

Now they appear strong exchange rings

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Descending chains in RCP(R)

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Theorem (Guil-Herzog)

If R is left cotorsion, then every descending chain in RCP(R) has a minimal lower bound.

But not all chains!

Just the compatible chains

Compatible descending chains in RCP(R)

Take a descending chain $\{\langle a_\alpha, b_\alpha\rangle \mid \alpha < \kappa\}$.

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$a_\alpha = a_\gamma r_{\gamma\alpha}$	$a_\beta = a_\alpha r_{\alpha\beta}$	$a_\beta = a_\gamma r_{\gamma\beta}$

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Notice

$a_\beta = a_\gamma r_{\gamma\alpha} r_{\alpha\beta}$ but not neccesarily $r_{\gamma\alpha} r_{\alpha\beta} = r_{\gamma\beta}$ for all $\gamma < \alpha < \beta$, when β is limit.

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Compatible descending chain

If there exist $\{r_{\alpha\beta} \mid \alpha < \beta < \kappa\}$ and $\{s_{\alpha\beta} \mid \alpha < \beta < \kappa\}$ with

- $a_\beta = a_\alpha r_{\alpha\beta}$.
- $r_{\gamma\beta} = r_{\gamma\alpha} r_{\alpha\beta}$.
- $b_\beta = b_\alpha s_{\alpha\beta}$.
- $s_{\gamma\beta} = s_{\gamma\alpha} s_{\alpha\beta}$.

Definition

R is right strong exchange if every compatible descending chain of right coprime pairs has a minimal lower bound.

Examples of right strong exchange rings

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The following are right strong exchange rings:

1. Left cotorsion, in particular, left self-injective and left pure-injective rings.
2. Local rings.
3. Left continuous rings.

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The following modules have right strong exchange endomorphism rings:

1. Cotorsion, in particular, injective and pure-injective.
2. Modules with local endomorphism ring.
3. Continuous.

Theorem (Cortés Izurdiaga-Guil, [1])

Every right strong exchange ring is semiregular, i. e., if J is the Jacobson radical of R,

- R/J is von Neumann regular.
- Idempotents lift modulo J

They are exchange

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3. Right strong exchange are not left strong exchange.

The idea

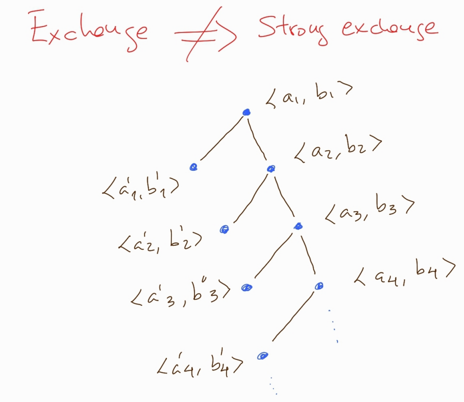

Figure 1: Exchange does not implies strong exchange

The full exchange property

The finite exchange property

Theorem (Warfield)

Modules with exchange endomorphism satisfy the finite exchange property.

The finite exchange property (Crawley-Jónsson, 1964)

M satisfies the finite exchange property if for any decompostion

$$X = M' \oplus Y = \bigoplus_{i=1}^{n} N_i$$

with $M' \cong M$, there exists $N_i' \leq N_i$ with

$$X = M' \bigoplus \Big(\bigoplus_{i=1}^{n} N_i' \Big)$$

The full exchange property (Crawley-Jónsson, 1964)

The full exchange property

M satisfies the full exchange property if for any decompostion

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Open problem (Crawley-Jónsson, 1964)

Does the finite exchange property imply the full exchange property?

Flat cotorsion modules

The endomorphism ring S of a flat cotorsion module M is right strong exchange.

$\Rightarrow$ S is exchange.

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Question 2

Has a flat cotorsion module the full exchange property?

Summable families

A family of endomorphisms $\{u_i \mid i \in I\}$ of M is summable if for any $m \in M$, the set

$$\{i \in I \mid (m)u_i \neq 0\} \text{ is finite.}$$

We can consider $\sum_{i \in I} u_i$!

Theorem (Huisge Zimmermann-Zimmermann)

The following are equivalent for a module M with $S = \mathrm{End}_R(M)$

1. M has the exchange property.
2. For any summanble family with $\sum_{i \in I} u_i = 1$, there exists a summable family of orthogonal idempotents, $\{e_i \mid i \in I\}$ with $\sum_{i \in I} e_i = 1$ and $e_i S \leq u_i S$.

Right coprime families

Right coprime family $\langle u_i \rangle_{i \in I}$ in the endomorphism ring S of M is a summable family $\{u_i \mid i \in I\}$ for which there exists a family of endomorphisms $\{a_i \mid i \in I\}$ satisfying

$$\sum_{i \in I} u_i a_i = 1$$

1. **Equivalence relation**:

$$\langle u_i \rangle_{i \in I} \sim \langle v_i \rangle_{i \in I} \Leftrightarrow u_i S = v_i S \qquad \forall i \in I$$

2. **Preorder relation**:

$$\langle u_i \rangle_{i \in I} \leq \langle v_i \rangle_{i \in I} \Leftrightarrow u_i S \leq v_i S \qquad \forall i \in I$$

Extension of RCP(S)

We have the partially ordered set $(\mathrm{RCF}_I(S), \leq)$.

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Notice: $\langle e_i \rangle_{i\in I} \leq \langle u_i \rangle_{i\in I}$ in RCF(S)

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Are the coprime families consisting of orthogonal idempotents the minimal elements in $\mathrm{RCF}(S)$?

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Answer

Yes!

Minimal elements in RCF(S)

Theorem (Cortés Izurdiaga-Guil-Srivastava, [2])

The following assertions are equivalent for $\langle u_i \rangle_{i \in I} \in \mathrm{RCF}(S)$:

1. $\langle u_i \rangle_{i \in I}$ is a minimal element in $\mathrm{RCF}(S)$.
2. There exists a family of orthogonal idempotents $\{e_i \mid i \in I\}$ such that $\langle e_i \rangle_{i \in I} = \langle u_i \rangle_{i \in I}$.

Minimal elements in $\mathrm{RCF}(S)$

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Theorem (Cortés Izurdiaga-Guil-Srivastava, [2])

If M is flat cotorsion with endomorphism ring R, then:

1. For any I and any $\langle u_i \rangle_{i \in I} \in \mathrm{RCF}(S)$, there exists a minimal $\langle e_i \rangle_{i \in I} \in \mathrm{RCF}(S)$ with $\langle e_i \rangle_{i \in I} \leq \langle u_i \rangle_{i \in I}$.

Minimal elements in RCF(S)

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2. M satisfies the full exchange property.

If $\mathcal{S}$ is a small preadditive category, then:

Theorem

Mod–$\mathcal{S}$ is equivalent to the category of unitary modules over a ring without unit (but with enough idempotents).

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Theorem

Mod$-\mathcal{S}$ *is equivalent to the category of unitary modules over a ring without unit (but with enough idempotents).*

Pure-injectives in FAA categories

If C is a FAA category and C is pure-injective, then:

1. $End(C)$ is right strong exchange.
2. C has the full exchange property.

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Thank you very much!